# Balanced Magic Squares

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# What is a Magic Square ?

- By a magic square we mean a matrix, where the sum of the entries in all rows, columns and both main diagonals is the same. This constant will be called the magic sum.
- For example the following matrix is a 3 by 3 square

| | | |
|---|---|---|
| 4 | 9 | 2 |
| 3 | 5 | 7 |
| 8 | 1 | 6 |

# Natural Magic Square of order n

is a magic square such that the entries of the square are the natural numbers from 1 to n^2 and that all entries are distinct. In this case the magic sum is equal to n(n^2+1)/2. For example, the magic constant of natural magic squares of order 6 is 111.

# How many natural magic squares are there?

| order | number |
|---|---|
| 3 | 1*8=8 |
| 4 | 220*32=7040 |
| 5 | 64,826,306*32=2,202,441,792 |
| 6 | ? |

# www.netstaff.co.jp/msq/msqe.htm

This site contains the program msq6e.exe, which generates all natural magic squares of order 6. The execution time on a usual PC (Pentium4 machine working at 3GHz) will be at least 220,000 years.

The site (cf. [1]) estimates the number of magic squares of order 6 to be in

**(1.7712e19, 1.7796e19)**

with a probability of **99%**.

# More definitions

- By a pandiagonal magic square we mean that the magic square satisfy: all bent diagonals sum up to the magic sum.
- Four corner property of a magic square 6 by 6 means the sum of four corners of any inside 4 by 4 square sums up to 74.
- Balanced magic square means that

$$a_{11} + a_{61} + a_{16} + a_{66} = 2s$$

$$a_{22} + a_{25} + a_{52} + a_{55} = 2s$$

$$a_{33} + a_{43} + a_{34} + a_{44} = 2s$$

# An Example of FC-squares

| | | | | | |
|---|---|---|---|---|---|
| 13 | 30 | 14 | 3 | 34 | 17 |
| 9 | 27 | 19 | 26 | 10 | 20 |
| 16 | 28 | 2 | 1 | 33 | 31 |
| 23 | 6 | 36 | 35 | 4 | 7 |
| 18 | 15 | 11 | 21 | 22 | 24 |
| 32 | 5 | 29 | 25 | 8 | 12 |

# How is the program constructed?

It consists basically nested for loops, where each small letter is represented by a loop, e. g. we find this paragraph in the program

```
 for a:=1 to 34 do begin
     for e:=a+1 to 35 do begin
         for b:=e+1 to 36 do begin

b[21]:=74 - a - b - e;

if b[21]<1 or b[21]>36 then goto below1;
```

# Features of the program

- It is easily parallelizable. It can be split into several programs, each starts with a different value of a, b and e. The programs can bre run separately without communications.
- It uses the symmetries (proved by mathematical relations) to reduce the run time. The calculated number will be later multiplied with 16.

| | | | | | |
|---|---|---|---|---|---|
| $k_1$ | * | * | * | * | $l_1$ |
| * | $k_2$ | * | * | $l_2$ | * |
| * | * | $k_3$ | $l_3$ | * | * |
| * | * | $l_4$ | $k_4$ | * | * |
| * | $l_5$ | * | * | $k_5$ | * |
| $l_6$ | * | * | * | * | $k_6$ |

An unique pair of magic diagonals (uPD) is a gPD with

$$k_1 < l_1 < l_6, \quad k_1 < k_6 .$$

Each uPD represents 8 different gPDs. Each gPD can be transformed in a uPD by transposition and 90° rotations. A normalized pair of magic diagonals (nPD) is an uPD with

$$k_1 < l_2, \dots, l_5, \quad k_1 < k_2 < k_3, k_4, k_5, \quad k_3 < k_4 .$$

The total number of 6x6 nPDs is

159 626 931 * 5 400,

where 5400 represents the number of all possible permutations of the entries in each diagonal.

First subset: Magic squares with a centrally symmetric pair of diagonals with the following structure

$$k_i + k_{7-i} = 37, l_i + l_{7-i} = 37, \quad i = 1,2,3$$

Second subset: Magic squares with an axially symmetric pair of diagonals with the following structure

$$k_i + l_{7-i} = 37, \quad i = 1,2,3,4,5,6$$

# Main results

- We determined the property preserving transformations for the new class of squares. Besides the eight classical ones there are 24 transformations. When we combine them together we obtain 24*8 property preserving transformations. By using this knowledge we reduce the run time. This number appears in our calculations. We computed the number of balanced magic 6x6 square matrices with self-similar pairs of diagonals (PDs). It is

$$12661555266936 * 24 * 8.$$

Further, we computed the number of following type of matrices ·
matrices with centrally self-similar PDs: 7 411 859 115 784 * 24 * 8

· matrices with axially self-similar PDs: 5 249 696 151 152 * 24 * 8

· matrices with centrally symmetric PDs: 5 836 806 535 224 * 24 * 8

· matrices with axially symmetric PDs: 2 355 312 270 384 * 24 * 8

· axially symmetric square matrices: 705 251 529 216 * 24 * 8.

# Conclusion

In this paper we walked further steps in the direction of solving the problem of counting magic squares 6 b 6. The subset of four corner magic squares, which is a subset of balanced magic squares, was counted in [2]. The number of free variables for this type is 18, while this number is for balanced (res. general) magic squares 21 (res. 23). We counted some subsets of balanced squares using the symmetries, which reduced he amount of calculations. In order to count all balanced squares we need better computers or a larger computer network.

# References


- 1) www.trump.de/magic-squares
- 2) www.grogono.com/magic
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- [2] S. Al-Ashhab, “Even-order Magic Squares with Special Properties,” Open Problems in Mathematics and Computer Science, 2012, Vol. 5, No. 2.
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